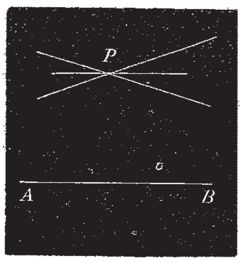
ON THE SIMPLEST CONTINUOUS MANIFOLDNESS OF TWO DIMENSIONS AND OF FINITE EXTENT[1]

ONE of the most remarkable speculations of the present century is the speculation that the axioms of geometry may be only approximately true, and that the actual properties of space may be somewhat different from those which we are in the habit of ascribing to it. It was Lobatchewsky who first worked out the conception of a space in which some of the ordinary laws of geometry should no longer hold good. Among the axioms which lie at the foundation of the Euclidian scheme, he assumed all to be true except the one which relates to parallel straight lines. An equivalent form of this axiom, and the one now generally employed in works on geometry, is the statement that it is impossible to draw more than one straight line parallel to a given straight line through a given point outside it. In other



words, if we take a fixed straight line, A B, prolonged infinitely in both directions, and a fixed point, P, outside it; then, if a second straight line, also infinitely prolonged in both directions, be made to rotate about P, there is *only one* position in which it will not intersect A B. Now Lobatchewsky made the supposition that this axiom should be untrue, and that there should be a finite angle through which the rotating line might be turned, without ever intersecting the fixed straight line, A B. And in following out the consequences of this assumption he was never brought into collision with any of the other axioms, but was able to construct a perfectly self-consistent scheme of propositions, all of them valid as analytical conceptions, but all of them perfectly incapable of being realised in thought.

Many of the results he arrived at were very curious; such as, for instance, that the three angles of a triangle would not be together equal to two right angles, but would be together less than two right angles by a quantity proportional to the area of the triangle. If we were to increase the sides of such a triangle, keeping them always in the same proportion, the angles would become continually smaller and smaller, until at last the three sides would cease to form a triangle, because they would never meet at all.

There are many other assumptions, at variance with the axioms of Euclid, which may be made respecting distance-relations, and which yield self-consistent schemes of propositions differing widely from the propositions of geometry. We see, therefore, that geometry is only a particular branch of a more general science, and that the conception of space is a particular variety of a wider and more general conception. This wider conception, of which time and space are particular varieties, it has been proposed to denote by the term *manifoldness*. Whenever a general notion is susceptible of a variety of specialisations, the aggregate of all such specialisations is called a manifoldness. Thus space is the aggregate of all *points*, and each point is a specialisation of the general notion of *position*. In the same way time is the aggregate of all *instants*, and each instant is a specialisation of the general notion of *position in time*. Space and time are, in fact, of all manifoldnesses, the ones with which we are by far the most frequently concerned.

[1] Read before the London Mathematical Society, December 14, 1876.

Now there is an important feature in which these two manifoldnesses agree. They are both of them of such a nature that no limit can be conceived to their divisibility. However near together two points in space may be, we can always conceive the existence of intermediate points. And the same thing holds in regard to time. Mathematicians express this fact by saying that space and time are *continuous* manifoldnesses. But there is another feature, equally important with the foregoing, in regard to which space and time are strikingly contrasted. If we wish to travel away from any particular instant in time, there are only two directions in which we can set out. We must either ascend or descend the stream. But from a point in space we can set out in an infinite number of directions. This difference is expressed by saying that time is a manifoldness of *one dimension*, and that space is a manifoldness of *more than one dimension*. An aggregate of points in which we could only travel backwards or forwards would be, *not* solid space, but a *line*. A line, therefore, is a manifoldness of one dimension. A *surface*, again, may be regarded as an aggregate of lines; and it is an aggregate of such a nature, that if we wish to travel away from a particular line, there are only two directions in which we can set out. It is therefore a line-aggregate of one dimension. Considered as a point-aggregate it has two dimensions, and accordingly it is a manifoldness of two dimensions. In the same way it will be seen that solid space is a manifoldness of three dimensions.

I have endeavoured by these remarks to explain what is meant when we speak of a continuous manifoldness of two dimensions. It is the object of this paper to communicate some results I have arrived at respecting the properties of the simplest of such manifoldnesses which has a finite extent. The existence of the particular manifoldness I shall endeavour to describe has been referred to in a remarkable lecture by Prof. Clifford on "The Postulates of the Science of Space," but, so far as I am aware, its properties have not hitherto been worked out in detail.

The simplest of all doubly-extended continuous manifoldnesses is the *plane*. But it is not a manifoldness of finite extent. It reaches to infinity in every direction, and its area is greater than any assignable area. It is therefore not the manifoldness of which we are in search. Now the circumstance in which the plane differs from those doubly-extended manifoldnesses which are next to it in order of simplicity, is the possibility that figures constructed in it may be magnified or diminished to any extent without alteration of shape; in other words, that figures which can be constructed in it at all can be constructed to any scale. That this property is not possessed by curved surfaces, may be seen by considering the case of a spherical triangle. If the sides of a triangle constructed on a given sphere be all of them increased or diminished in the same proportion, the shape of the triangle will not remain the same. Now it has been found by Prof. Riemann that this property of the plane is equivalent to the following two axioms:—(1) That two geodesic lines which diverge from a point will never intersect again, or, as Euclid puts it, that two straight lines cannot inclose a space; and (2) that two geodesic lines which do not intersect will make equal angles with every other geodesic line. The second is precisely equivalent to Euclid's twelfth axiom. Deny the first of these axioms, and you have a manifoldness of uniform positive curvature; deny the second, and you have one of uniform negative curvature. The plane lies midway between the two, and its curvature is zero at every point.

Let us consider, then, the case of a doubly-extended manifoldness, of which the curvature is uniform and positive. The first of the before-mentioned two axioms

is no longer true. Geodesic lines diverging from a point do not continue to diverge for ever. They meet again and inclose a space. The first question which presents itself is with reference to the situation of the point towards which they ultimately converge. In the case of a spherical surface they will converge towards a point which is separated from the starting-point by half the length of a geodesic line. And this is the only case we are able to conceive. The surface of a sphere is the only doubly extended manifoldness of uniform positive curvature which geometry recognises, and it is the only one which we can figure to ourselves in thought. It is not, however, the *simplest* of such manifoldnesses. To obtain the simplest case we must suppose that the point towards which two geodesic lines converge is separated from their starting-point, not by *half*, but by the *entire* length of a geodesic line; or, what amounts to the same thing, that it *coincides* with the starting-point. It is true that we are utterly unable to figure to ourselves a surface in which two geodesic lines shall have only one point of intersection, and shall yet inclose a space. But we are perfectly at liberty to reason about such a surface, because there is nothing self-contradictory in the definition of it, and because, therefore, the analytical conception of it is perfectly valid. It is the simplest continuous manifoldness of two dimensions, and of finite extent, and those few properties of it which I have worked out appear to me to be very beautiful. In order to make my observations more intelligible, I shall for the future speak of it as a surface, and its geodesic lines I shall speak of as straight lines. I have the highest authority for using this nomenclature, and though it will impart to my theorems a very paradoxical sound, it is calculated, I think, to give a juster idea of their meaning, than if I were to use the more accurate, but less familiar terms.

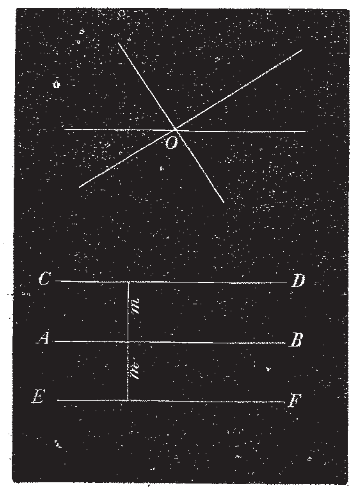
Assuming, then, as the fundamental properties of our surface, that every straight line is of finite extent (in other words, that a point moving along it will arrive at the position from which it started after travelling a finite distance), and that two straight lines cannot have two points in common, the first corollary I propose to establish is that all straight lines in the surface are of equal extent.

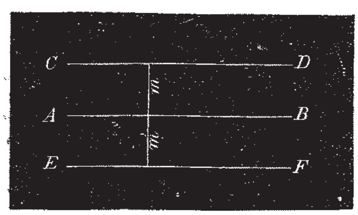
Let A, B, be two straight lines in the given surface. If possible, let A be greater than B. From A cut off a portion equal to B. Let P, Q, be the extreme points of this segment, and let R be any point in B. Apply the line A to the line B in such a manner that the point P falls on the point R, then, since in a surface of uniform curvature equal lengths of geodesic lines may be made to coincide, the segment PQ will coincide with the entire straight line B. Hence Q will fall upon R. But P coincides with R, and P and Q do not coincide with one another, since PQ is less than the entire straight line A; therefore Q cannot coincide with R. Hence A cannot be greater than B.

The straight lines here spoken of are, of course, not *terminated* straight lines. What the proposition asserts is that the *entire* length of all straight lines in the given surface is the same. The corresponding proposition in spherical geometry is that all great circles of a given sphere are equal.

There are a great many other analogies between the imaginary surface here treated of and the surface of a sphere. Its straight lines, though they are like the straight lines of a plane in the circumstance that any two of them have only one point of intersection, are in many other respects analogous to great circles. In any of its straight lines, for instance, each point has a corresponding point which is *opposite* to it, and farther from it than any other point in the line. For if by setting out from a point and travelling a finite distance in a particular direction we get back to the starting-point, there must be a point half way on our journey which is farther from the starting-point than any other point in the line, and which may very appropriately be called its opposite point. It is an obvious corollary that the distance between any two points will be the same as the distance between their opposite points.

Let us now consider the case of a number of straight lines radiating from a centre. In each of them there will be a point which is opposite to that centre. And it will be a separate point for every separate straight line. For no two straight lines can have two points in common, and since these radiating lines have a common centre of radiation, they can have no other point in common. Hence, if we suppose one of these lines to rotate about the centre, the point opposite to the centre will describe a continuous line, and one which finally returns into itself. It is the locus of all points in the surface opposite to the centre of radiation. What now is the character of this locus? In the first place it is a line which is of the same shape all along, and of which all equal segments therefore can be made to coincide. For any two positions of





the rotating line which contain a given angle may be placed upon any other two positions which contain an equal angle. Then, since the length of all straight lines in the surface is the same, the opposite points will coincide, and by parity of reasoning all intermediate points of the locus. But, in the second place, the locus is also of the same shape on both sides. For each point in it may be approached from the centre of radiation in two different ways, and it is at the same distance from that centre, whether it be approached in the one way or the other. Any particular segment, in fact, of the locus has its extreme points joined to the centre of radiation by lines which are of equal length, and which include an equal angle—lines, therefore, which may be made to coincide. Since this is the case for any segment whatever, and for every subdivision of a segment, all the points of a segment will still remain on it if the segment be turned round and applied to itself. Hence the locus is of the same shape, whether viewed from the one side or from the other. But since it is also of the

same shape all along, it satisfies Leibnitz's definition of a straight line, and it is, in fact, a geodesic line of the surface.

Hence we have this second proposition—that all points in the surface opposite to a given point lie in a straight line.

From the method of its construction, this straight line is farther from the given point than any other line in the surface. Travelling from the given point as a centre, in whatever direction we might set out, we should, after completing half our journey, arrive at this farthest straight line, we should cross it at right angles, and we should then keep getting nearer and nearer to our starting-point, until we finally reached it from the opposite side.

Each separate point in the surface, moreover, has a separate farthest line. For if any two points be taken, the points opposite to them on the straight line which joins them will be distinct. Hence their farthest lines will cut this joining line in two separate points. They must, therefore, be two *separate* lines, for the same straight line cannot cut another straight line in two separate points. In a similar manner it may be shown that each straight line in the surface has a separate farthest point. Hence there exists a reciprocal relation between the points and straight lines of the surface, a relation which we may express by saying that every point in the surface has a *polar*, and that every straight line in the surface has a *pole*. It is then easy to show that when a point is made to move along a straight line its polar will turn about a point, and that when a straight line is made to turn about a point, its pole will move along a straight line.

It is interesting to compare these propositions with the corresponding ones in spherical geometry. There, too, each point has a farthest geodesic line; that is to say, a geodesic line which is farther from it than any other geodesic line on the sphere. But each geodesic line has *two* farthest points or poles, instead of having only one. Hence there is not that perfect reciprocity of relationship between points and geodesic lines which exists in the surface we have been examining; and this is one of the many ways in which the sphere shows itself to be inferior to that surface in simplicity.

The most astounding fact I have elicited in connection with this surface is one which comes out in the theory of the circle. Defining a circle as the locus of points equidistant from a given point, we shall find that it assumes a very extraordinary shape when its radius is at all nearly equal to half the entire length of a straight line. For let us again figure to ourselves a number of straight lines radiating from a point. Let l be the total length of each straight line. Then the supposition we have to make is that the radius of our circle shall be nearly equal to $\frac{l}{2}$. Let us suppose it equal to $\frac{l}{2} - m$, where m is small as compared with l. Each of the radiating lines will cut the circle in two points, and each of these points will be at a distance from O equal to $\frac{l}{2} - m$ or $\frac{l}{2} + m$, according as the distance is measured in the one direction or the other. And their distance from each other will be equal to $2m$, that is to say, it will be comparatively small. But each point on the polar of O will be at a distance from O equal to $\frac{l}{2}$. Hence each point on the circle will be at a distance from this polar equal to m. Moreover, every point at a distance of m from the polar will be a point on the circle, because it will be at a distance of $\frac{l}{2} - m$ from O. But the locus of points at a distance of m from the straight line, AB, will consist of two branches, CD and EF, one on either side of AB, and at the same distance from it along their whole length. It is true that these branches form in reality a single continuous line. A point travelling along from C to D, and further in the same direction, would ultimately appear at E, travel along to F, and then, after a further journey, reappear at the point C. But this does not alter the fact that when a small portion only of this line is contemplated, it presents the appearance of two straight lines, each of them parallel to, and equidistant from, AB.

In the limiting case, where the radius becomes equal to $\frac{l}{2}$, CD and EF both of them coincide with AB. The circle merges into a straight line, and becomes, in fact, the polar of its own centre. It is not, indeed, quite accurate to say that it merges into a straight line, for it reduces itself rather to two coincident straight lines, and its equation in co-ordinate geometry would be one of the second degree.

In regard to the surface here treated of, it is easy to see that, as with the sphere, the smaller the portion of it we bring under our consideration, the more nearly its properties approach to those of the plane. Indeed, if we consider an area that is very small as compared with the total area of the surface, its properties will not differ sensibly from those of the plane. And on this ground it has been argued that the universe may in reality be of finite extent, and that each of its geodesic lines may return into itself, provided only that its total magnitude be very great as compared with any magnitude which we can bring under our observation.

In conclusion, I cannot do better than quote the passage in which Prof. Clifford explains what must be the constitution of space if this hypothesis should be true. "In this case," he says, "the universe, as known, is again a valid conception, for the extent of space is a finite number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever and move in that direction in a perfect straight line according to the definition of Leibnitz, after travelling a most prodigious distance, to which the parallactic unit—200,000 times the diameter of the earth's orbit—would be only a few steps you would arrive at—this place. Only, if you had started upwards, you would appear from below. Now one of two things would be true. Either when you had got half way on your journey you came to a place that is opposite to this, and which you must have gone through, whatever direction you started in, or else all paths you could have taken diverge entirely from each other till they meet again at this place. In the former case every two straight lines in a plane meet in two points, in the latter they meet only in one. Upon this supposition of a positive curvature the whole of geometry is far more complete and interesting; the principle of duality, instead of half breaking down over metric relations, applies to all propositions without exception. In fact I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."

F. W. FRANKLAND

HYDROGRAPHY OF WEST CENTRAL AFRICA

MR. STANLEY'S second letter in last Thursday's *Telegraph* contains important information on the district between Tanganyika and the Albert and Victoria Nyanza—information complementary to that given in his former letters, which we embodied in a map, vol. xiv. p. 374. He has, in fact, discovered another "source" of the Nile, and one evidently of great length and volume—the Kagera—which he has gallantly named the Alexandra Nile. This river issues from a large lake, Akanyaru or Alexandra Nyanza, in two branches and flows north, uniting under 1° S. lat., and flowing east to the Victoria Nyanza. Mr. Stanley was only able to see the Alexandra Nyanza from a distance, but it is evidently of consider-